AN ELEMENTARY EVALUATION OF $\zeta(2n)$ USING DIRICHLET'S KERNEL

MICAH B. MILINOVICH AND UNIQUE SUBEDI

We use Dirichlet's kernel to give a simple proof of the classical identity of $\zeta(2n)$. Our proof simplifies the proof in [3] and shows a deeper connection between Bernoulli numbers and Dirichlet's kernel.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta function. The series representation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely for $s \in \mathbb{C}$ when $\mathrm{Re}(s) > 1$. For $n \in \mathbb{N}$, we define Bernoulli polynomials $B_n(x)$ by the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \tag{1.1}$$

for $|z| < 2\pi$. We call $B_n(0)$ the n^{th} Bernoulli number, which henceforth will be denoted as B_n. Dirichlet's kernel is defined by

$$D_n(x) := \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \cos kx = \frac{\sin((n+1/2)x)}{\sin(x/2)}. \tag{1.2}$$

In this paper, we use Dirichlet's kernel to prove the following classical result

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} \tag{1.3}$$

for $n \in \mathbb{N}$.

Our work is motivated by the elegant calculation of [2], which uses Dirichlet's kernel to give a quick proof of the identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Stark establishes this identity by evaluating the integral

$$\int_0^{\pi} t\, D_{2m-1}(t)\, dt,$$

for $m \in \mathbb{N}$ in two different ways. On one hand, he evaluates this integral by using the definition of the Dirichlet kernel as the sum of cosines in (1.2). On the other hand, he evaluates (2.1) by expressing $D_{2m}(\pi t)$ as a ratio of sines. Upon letting $m \to \infty$, he derives the identity

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8},$$

which immediately gives the identity for $\zeta(2)$.

2. Evaluating $\zeta(2n)$

We prove (1.3) by evaluating the integral

$$\int_0^1 B_{2n}(t)\, D_m(2\pi t)\, dt, \tag{2.1}$$

for $m, n \in \mathbb{N}$, in two different ways. On one hand, we evaluate this integral by using the definition of the Dirichlet kernel as the sum of exponentials in (1.2). On the other hand, we evaluate (2.1) by expressing $D_{2m}(2\pi t)$ as a ratio of sines. The formula for $\zeta(2n)$ in (1.3) will follow from these two calculations upon letting $m \to \infty$.

2.1. Properties of Bernoulli polynomials and Bernoulli numbers.

We now state the properties of Bernoulli polynomials and Bernoulli numbers necessary for our proof. For $n \in \mathbb{N}$, we recall the well-known identities

$$B_n'(t) = n\, B_{n-1}(t) \tag{2.2}$$

and

$$\int_0^1 B_n(x)\, dx = 0. \tag{2.3}$$

Also recall that $B_1(1) = -B_1(0) = \frac{1}{2}$ and $B_n(0) = B_n(1)$ for $n \geq 1$. Standard properties of Bernoulli polynomials and numbers can be found in [1, Appendix B].

The following integral is a key component of our proof of (1.3).

Lemma 1. *For $k \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{N}$, we have*

$$\int_0^1 B_{2n}(t)\, e^{2\pi i k t}\, dt = \frac{(-1)^{n-1}(2n)!}{(2\pi k)^{2n}}.$$

Proof. We use (2.2) to integrate by parts $2n-1$ times. Since $B_n(0) = B_n(1)$ for $n \geq 2$ and $e^{2\pi i k t} = 1$ at both end points for all $k \in \mathbb{Z}$, all but one term vanishes leaving

$$\frac{(-1)^{2n-1}(2n)!}{(2\pi i k)^{2n-1}} \int_0^1 B_1(t)\, e^{2\pi i k t}\, dt.$$

Integrating by parts again and using the facts $B_1(1) = -B_1(0) = \frac{1}{2}$ and $\int_0^1 e^{2\pi i k t}\, dt = 0$, this reduces to

$$\frac{(-1)^{2n-1}(2n)!}{(2\pi i k)^{2n}}.$$

The lemma now follows upon using $i^{2n} = (-1)^n$. $\square$

2.2. Evaluating the integral with $D_m(2\pi t)$ as a sum of exponentials.

Using the first representation for the Dirichlet kernel in (1.2) and then interchanging the order of integration and summation, we derive that

$$\begin{aligned}\int_0^1 B_{2n}(t)\, D_m(2\pi t)\, dt &= \int_0^1 B_{2n}(t) \left(\sum_{k=-m}^{m} e^{2\pi i k t} \right) dt \\ &= \int_0^1 B_{2n}(t)\, dt + \sum_{\substack{k=-m \\ k \neq 0}}^{m} \int_0^1 B_{2n}(t) e^{2\pi i k t}\, dt\end{aligned}$$

By (2.3), the integral of the Bernoulli polynomial vanishes. In the remaining sum over even k, we use Lemma 1 to find that

$$\begin{aligned}\int_0^1 B_{2n}(t)\, D_m(2\pi t)\, dt &= \frac{(-1)^{n-1}(2n)!}{(2\pi)^{2n}} \sum_{\substack{k=-m\\k\neq 0}}^{m} \frac{1}{k^{2n}} \\ &= 2\frac{(-1)^{n-1}(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{m} \frac{1}{k^{2n}}\end{aligned} \tag{2.4}$$

2.3. **Evaluating the integral with $D_m(2\pi t)$ as a ratio of sines.** For fixed n, our goal is to show that

$$\int_0^1 B_{2n}(t)\, D_m(2\pi t)\, dt = B_{2n} + O\Big(\frac{1}{m}\Big). \tag{2.5}$$

Since $B_{2n}(0) = B_{2n}(1) = B_{2n}$, it follows that

$$B_{2n}(t) = B_{2n} + \big(B_{2n}(t) - B_{2n}(0)\big) = B_{2n} + t\,(t-1)\, P_n(t)$$

for some polynomial $P_n(t)$. The first representation for the Dirichlet kernel in (1.2) implies that

$$\int_0^1 D_m(2\pi t)\, dt = \int_0^1 \Big(1 + \sum_{\substack{k=-m\\k\neq 0}}^{m} e^{2\pi i k t}\Big) dt = 1.$$

Hence, using the definition of $P_n(t)$ and the third representation for $D_m(2\pi t)$ in (1.2), we derive that

$$\begin{aligned}\int_0^1 B_{2n}(t)\, D_m(2\pi t)\, dt &= B_{2n} \int_0^1 D_m(2\pi t)\, dt + \int_0^1 t\,(t-1)\, P_n(t)\, D_m(2\pi t)\, dt \\ &= B_{2n} + \int_{0^+}^{1^-} t\,(t-1)\, P_n(t)\, \frac{\sin\left((2m+1)\pi t\right)}{\sin\left(\pi t\right)}\, dt,\end{aligned}$$

where 0^+ and 1^- indicate right-hand and left-hand limits as we approach 0 and 1 respectively. Integrating by parts, we find that

$$\begin{aligned}&\int_{0^+}^{1^-} t\,(t-1)\, P_n(t)\, \frac{\sin\left((2m+1)\pi t\right)}{\sin\left(\pi t\right)}\, dt \\ &\quad = \left(\frac{t\,(t-1)\, P_n(t)}{\sin\left(\pi t\right)}\right) \frac{\cos\left((2m+1)\pi t\right)}{\pi(2m+1)}\Bigg|_{0^+}^{1^-} - \int_{0^+}^{1^-} \frac{d}{dt}\left\{\frac{t\,(t-1)\, P_n(t)}{\sin\left(\pi t\right)}\right\} \frac{\cos\left((2m+1)\pi t\right)}{\pi(2m+1)}\, dt.\end{aligned}$$

Letting $f(t) = (t\,(t-1))/\sin(\pi t)$, this equals

$$f(t)\, P_n(t)\, \frac{\cos\left((4m+1)\pi t\right)}{\pi(4m+1)}\Bigg|_{0^+}^{1^-} - \int_{0^+}^{1^-} \big(f'(t)\, P_n(t) + f(t)\, P_n'(t)\big)\, \frac{\cos\left((4m+1)\pi t\right)}{\pi(4m+1)}\, dt.$$

A standard calculus exercise shows that

$$-\frac{1}{\pi} < f(t) < -\frac{1}{4} \quad \text{and} \quad -\frac{1}{\pi} < f'(t) < \frac{1}{\pi}$$

for $0 < t < 1$. Thus, recalling that n fixed, we conclude that

$$\int_{0^+}^{1} t\,(t-1)\, P_n(t)\, \frac{\sin\left((4m+1)\pi t/2\right)}{\sin\left(\pi t/2\right)}\, dt = O\Big(\frac{1}{m}\Big).$$

Combining estimates, we have proved (2.5).

2.4. **Finishing the proof.** Equating the expressions in (2.4) and (2.5), we have shown that

$$2\frac{(-1)^{n-1}(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{m}\frac{1}{k^{2n}} = B_{2n} + O\Big(\frac{1}{m}\Big).$$

Letting $m \to \infty$, we now see that

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}$$

for every $n \in \mathbb{N}$.

References

[1] Montgomery, H. L., and Vaughan, R. C. *Multiplicative Number Theory I: Classical Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.

[2] Stark, E. L. Application of a mean value theorem for integrals to series summation. *The American Mathematical Monthly 85*, 6 (1978), 481–483.

[3] Óscar Ciaurri, Navas, L. M., Ruiz, F. J., and Varona, J. L. A simple computation of (2k). *The American Mathematical Monthly 122*, 5 (2015), 444–451.