ON L_1-NORM OF DIRICHLET'S KERNEL

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1. Introduction

Dirichlet's kernel is defined by

$$D_n(x) := \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \cos kx = \frac{\sin((n+1/2)x)}{\sin(x/2)}. \tag{1.1}$$

$D_n(x)$ is periodic (mod π). So, its L_1 norm is defined as

$$||D_n||_1 := \frac{1}{\pi}\int_0^{\pi} |D_n(x)|\, dx.$$

It is well-known [1, Section 8.3] that

$$||D_n||_1 = \frac{4}{\pi^2}\log n + O(1) \text{ for } n \geq 1.$$

However, it turns out that we can obtain the closed-form expression of L_1-norm of Dirichlet's kernel, which allows us to get the following refined estimate.

Theorem 1. *Let γ be Euler-Mascheroni constant. For $n \in \mathbb{N}$, we have*

$$||D_n||_1 = \frac{4}{\pi^2}\log(2n+1) + C + O\Big(\frac{1}{n^2}\Big)$$

where

$$C := \frac{4}{\pi^2}\Big(\sum_{k=1}^{\infty} \frac{2\log k}{(4k^2-1)} + \log 4 + \gamma\Big) \tag{1.2}$$

We deduce Theorem 1 from the following exact formula for $||D_n||_1$.

Theorem 2. *For $n \in \mathbb{N}$, we have*

$$||D_n||_1 = \frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sum_{\ell=1}^{k(2n+1)}\frac{1}{2\ell-1}.$$

Using Euler-Maclaurin summation, we derive the following corollary.

Corollary 3. *Let B_m is m^{th} Bernoulli number. For $n, M \in \mathbb{N}$,*

$$||D_n||_1 = \frac{4}{\pi^2}\log(2n+1) + C + \sum_{m=1}^{M-1}\frac{C_m}{(2n+1)^{2m}} + O\Big(\frac{B_{2M}}{Mn^{2M}}\Big)$$

where C is defined on (1.2), and

$$C_m := \frac{4B_{2m}(1-2^{1-2m})}{m\pi^2}\sum_{k=1}^{\infty}\frac{1}{(4k^2-1)k^{2m}}, \text{ for } m \geq 1. \tag{1.3}$$

Remark: Theorem 1 corresponds to the case $M = 1$.

We use the following lemmas to prove theorem 2.

2. Lemmas

Lemma 4. *The Fourier series of* $|\sin x|$ *on* $(-\pi,\pi)$ *is*

$$|\sin x| = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{(\sin kx)^2}{4k^2-1}.$$

Proof. The function $|\sin x|$ is even and has periodicity π, thus Fourier series of $|\sin x|$ has a form

$$|\sin x| = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(2kx),$$

where

$$a_k := \frac{2}{\pi}\int_0^{\pi} \sin x \cos(2kx)\, dx = \frac{1}{\pi}\int_0^{\pi}\Big(\sin((2k+1)x + \sin((1-2k)x\Big)\, dx = \frac{2}{\pi}\Big(\frac{1}{2k+1} - \frac{1}{2k-1}\Big)$$
$$= \frac{-4}{\pi(4k^2-1)}.$$

So, we obtain

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1}.$$

Since

$$\sum_{k=1}^{\infty}\frac{1}{4k^2-1} = \frac{1}{2}\Big(\sum_{k=1}^{\infty}\frac{1}{2k-1} - \frac{1}{2k+1}\Big) = \frac{1}{2},$$

we can write

$$|\sin x| = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2kx)}{4k^2-1} = \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{(\sin kx)^2}{4k^2-1}.$$

□

Lemma 5. *For any* $k \in \mathbb{N}$*, we have*

$$\int_0^{\pi/2}\frac{\sin^2 ky}{\sin y} = \sum_{\ell=1}^{k}\frac{1}{2\ell-1}.$$

Proof. The proof relies on the trigonometric identity

$$\sin^2 x - \sin^2 y = \sin(x+y)\sin(x-y) \text{ for all } x, y \in \mathbb{R}.$$

Define

$$I_k := \int_0^{\pi/2}\frac{\sin^2 ky}{\sin y}\, dy.$$

Then, we get the recursive identity

$$I_{k+1} - I_k = \int_0^{\pi/2}\frac{\sin^2(k+1)y - \sin^2 ky}{\sin y}\, dy = \int_0^{\pi/2}\sin((2k+1)y)\, dy = \frac{1}{2k+1}.$$

Using

$$I_1 = \int_0^{\pi/2}\frac{\sin^2 y}{\sin y}\, dy = 1$$

and the recursive identity, our claim follows by the induction on n. □

Lemma 6. *Let γ be Euler-Mascheroni constant. For $N, M \in \mathbb{N}$,*

$$\sum_{\ell=1}^{N} \frac{1}{2\ell-1} = \frac{\log N}{2} + \frac{\log 4+\gamma}{2} + \sum_{m=1}^{M-1} \frac{B_{2m}(2^{1-2m}-1)}{4m\, N^{2m}} + O\Big(\frac{B_{2M}}{MN^{2M}}\Big)$$

where B_k is the k^{th} Bernoulli's number and the implied constant is independent of M, N.

Proof. Note that

$$\sum_{\ell=1}^{N} \frac{1}{2\ell-1} = \left(\sum_{\ell=1}^{N} \frac{1}{2\ell-1} + \sum_{\ell=1}^{N} \frac{1}{2\ell}\right) - \sum_{\ell=1}^{n} \frac{1}{2\ell} = \sum_{\ell=1}^{2N} \frac{1}{\ell} - \frac{1}{2}\sum_{\ell=1}^{N} \frac{1}{\ell} \tag{2.1}$$

Standard application of Euler-Maclaurin Summation gives

$$\sum_{\ell=1}^{N} \frac{1}{\ell} = \log N + \gamma + \frac{1}{2N} + \sum_{m=1}^{M-1} \frac{B_{2m}}{2mN^{2m}} + O\Big(\frac{B_{2M}}{MN^{2M}}\Big).$$

Our desired result follows upon employing the previous estimate in (2.1). □

3. Proof of Theorem 2

Proof. From the last equality in (1.1), we may write L_1 norm of Dirichlet's kernel as

$$||D_n||_1 = \frac{1}{\pi}\int_0^{\pi} \frac{|\sin((n+1/2)x)|}{\sin(x/2)}\, dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin((2n+1)y)|}{\sin(y)}\, dy,$$

where the second equality follows from the change of variables. Employing the Fourier series of $|\sin((2n+1)y)|$ from Lemma 4, we get

$$\frac{16}{\pi^2}\int_0^{\pi/2} \sum_{k=1}^{\infty} \frac{(\sin(k(2n+1)y))^2}{(4k^2-1)\sin y}\, dy,$$

which can be further expressed as

$$\frac{16}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{4k^2-1}\int_0^{\pi/2} \frac{(\sin((2n+1)y))^2}{\sin y}\, dy$$

by switching order of integration and summation using Fubini's Theorem. Finally, the application of Lemma 5 gives the desired formula

$$\frac{16}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{4k^2-1}\sum_{\ell=1}^{k(2n+1)} \frac{1}{2\ell-1}.$$

□

4. Proof of Corollary 3

Proof. Substituting the result of Lemma 6 in the result of Theorem 2, we get

$$||D_n||_1 = \frac{16}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{4k^2-1}\left(\frac{\log(k(2n+1))}{2} + \frac{\log 4+\gamma}{2} + \sum_{m=1}^{M-1} \frac{B_{2m}(2^{1-2m}-1)}{4m(k(2n+1))^{2m}} + O\Big(\frac{B_{2M}}{M((kn)^{2M})}\Big)\right),$$

which can be rearranged as

$$\frac{4}{\pi^2}\log(2n+1) + \frac{4}{\pi^2}\Big(\sum_{k=1}^{\infty} \frac{\log k}{4k^2-1} + \log 4 + \gamma\Big) + \frac{4}{\pi^2}\sum_{m=1}^{M}\sum_{k=1}^{\infty} \frac{B_{2m}(2^{1-2m}-1)}{m(4k^2-1)k^{2m}(2n+1)^{2m}} + O\Big(\frac{B_{2M}}{Mn^{2M}}\Big).$$

With definition of C and C_m stated in (1.2) and (1.3) respectively, we get the form,

$$||D_n||_1 = \frac{4}{\pi^2}\log(2n+1) + C + \sum_{m=1}^{M-1}\frac{C_m}{(2n+1)^{2m}} + O\Big(\frac{B_{2M}}{Mn^{2M}}\Big),$$

as expressed in Corollary 3. □

References

[1] Zygmund, A. *Trigonometrical series.* 1935.