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General Section

A weighted version of the Erdős–Kac theorem

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ABSTRACT

Let $\omega(n)$ denote the number of distinct prime factors of n and let $\tau_k(n)$ denote the k-fold divisor function. Adapting a method of Granville and Soundararajan, we evaluate the centralized moments of $\omega(n)$, weighted by $\tau_k(n)$, and deduce a weighted version of the Erdős–Kac Theorem.

1. Introduction

Let $\omega(n)$ denote the number of distinct prime factors of natural number n. That is,

$$\omega(n) = \sum_{p|n} 1.$$

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The celebrated Erdős–Kac theorem [9] from 1940 states that for each $\alpha \in \mathbb{R}$, we have

$$\frac{1}{x} \sum_{\substack{n \leq x \\ \omega(n) - \log\log x \leq \alpha\sqrt{\log\log x}}} 1 \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt \tag{1.1}$$

as $x \to \infty$. In other words, since the $\log\log$ function grows very slowly, the quantity

$$\frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \tag{1.2}$$

follows a Gaussian distribution law with mean 0 and variance 1. Rényi and Turán [19] provided a (best possible) quantitative version of the theorem with $O(1/\sqrt{\log\log x})$ as the rate of convergence in (1.1).

Although the Erdős–Kac theorem is a classical result, it has remained a source of great research interest even in modern times. Many notable mathematicians have revisited the theorem and provided different proofs of it, e.g. [9,16,19,5,13,2,12,14]. We highlight a couple of these approaches in particular. Generalizing the prime counting function $\pi(x)$, let $\pi_k(x)$ denote the number of integers $n \leq x$ with $\omega(n) = k$. One approach to the Erdős–Kac theorem, related to the proof of Rényi and Turán [19] and the Selberg–Delange Method [18, Chapter 7.4], is to use asymptotics for $\pi_k(x)$ with $k \leq \log\log x + \alpha\sqrt{\log\log x}$ to evaluate the left hand side of (1.1). Such a proof is rather involved, requiring complex analysis and the theory of the Riemann zeta function, and it is at least as deep as the prime number theorem. Another approach is to asymptotically evaluate the moments of the quantity (1.2) and show that they match the moments of a standard Gaussian random variable. Since a Gaussian distribution is completely characterized by its moments [3, Theorem 30.1], this implies the Erdős–Kac Theorem. The moments approach was first accomplished by Delange [5] in 1953 and Halberstam [13] in 1955, although their proofs were rather complicated. About a decade later, Billingsley [2] gave a much simpler demonstration. In 2007, Granville and Soundararajan [12] provided an even more transparent and flexible treatment with their sieve method (see also [1]). Their approach remains one of the most direct and elementary ways to prove the Erdős–Kac theorem.

This paper is concerned with a generalization of the Erdős–Kac theorem in which the distribution of $\omega(n)$ is studied, but counting is weighted by the divisor function. In 2015, Elliott [7,8] proved that[1]

$$\Big(\sum_{n \leq x} \tau(n)\Big)^{-1} \sum_{\substack{n \leq x \\ \omega(n) - 2\log\log x \leq \alpha\sqrt{2\log\log x}}} \tau(n) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt, \tag{1.3}$$

[1] Elliott more generally proved (1.3) with $\tau(n)^{\alpha}$ for any $\alpha > 0$.

where $\tau(n) = \sum_{d|n} 1$ is the divisor function. Thus, when weighted by the divisor function, $\omega(n)$ still follows a Gaussian distribution as in the Erdős–Kac theorem, but with double the mean and double the variance. Such a result can be predicted by the following heuristic. Recall that $\tau(n) = 2^{\omega(n)}$ for square-free n. So, roughly speaking, we are studying the Gaussian distributed $\omega(n)$, tilted by its exponential. Consider a Gaussian random variable with mean 0 and variance 1, so that its distribution function is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-x^2}{2}} dx.$$

If we weight the measure by e^x, the distribution function becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-x^2}{2}} e^x dx = \frac{\sqrt{e}}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-(x-1)^2}{2}} dx,$$

where equality is obtained by completing the square. Thus, the resulting distribution with the weighted measure is still Gaussian but with altered mean and variance. In probability theory, this phenomenon is a simple case of Girsanov's Theorem [15, Chapter 3.5]. Elliott's result in (1.3) was generalized to short interval sums $\sum_{x \le n \le x+y}$, where y is a small power of x, by Liu and Wu [17], using similar methods. More general weighted Erdős–Kac type theorems, allowing for instance multiplicative weights other than $d(n)$, have been proven[2] by Elboim and Gorodetsky [6] and Tenenbaum [21,22].

Weighted central limit theorems are also of current interest in other parts of number theory. For instance, there is a famous and classical result of Selberg which establishes that $\log |\zeta(\frac{1}{2} + it)|$ has a Gaussian distribution for $t \in [T, 2T]$ as $T \to \infty$, where $\zeta(s)$ denotes the Riemann zeta-function. Recently, Fazzari [10,11] has proved weighted versions of Selberg's central limit theorem assuming the Riemann hypothesis.

Elliott's proof of the weighted Erdős–Kac theorem is based on the Selberg–Delange approach. Granted a trivial modification, it yields convergence to the normal law in a more general case than ours since it does not need the parameter 2^α (corresponding to our k, see below) to be integral. The proof of Elboim and Gorodetsky is based on moments and the method of Billingsley. The proof of Tenenbaum uses characteristic functions and new results on averages of multiplicative functions, of Wirsing and Halász type. Its framework is very general, essentially requiring that the non-negative, multiplicative weight is bounded above on primes and bounded below on large primes. Moreover, it furnishes an essentially optimal estimate for the speed of convergence. The goal of this paper is to give a relatively simple proof of Elliott's result (1.3) for the k-fold divisor function, using an approach based on moments and the sieve method of Granville and Soundararajan. Having such an approach to this weighted problem is natural, given the historical development of the Erdős–Kac theorem.

2 Gorodetsky kindly informed us of the work in [6,21,22] after the initial submission of this paper.

Thus, we compute the centralized moments of $\omega(n)$ weighted by $\tau_k(n)$ by generalizing the method of Granville and Soundararajan [12]. Although the foundation of our approach was laid out elegantly in [12], our proof requires a number of nontrivial modifications. It was not clear, a priori, whether the Granville-Soundararajan approach would work in this case and our generalization requires a careful set-up. We now state our main theorem.

Let

$$\tau_k(n) = \sum_{n_1 \cdots n_k = n} 1$$

be the k-fold divisor function. In Section 2.2, we show that, as n ranges over the integers below x, the mean of $\omega(n)$ with respect to the weighted measure $\tau_k(n)$ is asymptotic to $k \log\log x$. Thus we centralize by $k \log\log x$ and evaluate the following m-th moment for $(\omega(n) - k \log\log x)$, weighted by $\tau_k(n)$.

Theorem 1.1. *Let k and m be fixed natural numbers. For $x > 3$, we have*

$$\frac{\sum_{n \le x} \big(\omega(n) - k \log\log x\big)^m \tau_k(n)}{\sum_{n \le x} \tau_k(n)} = \begin{cases} (m-1)!! \big(k \log\log x\big)^{m/2} + O\Big((\log\log x)^{\frac{m-1}{2}}\Big), & \text{if } m \text{ is even}, \\ O\big((\log\log x)^{\frac{m-1}{2}}\big), & \text{if } m \text{ is odd}, \end{cases} \tag{1.4}$$

where $(m-1)!!$ denotes the product of all odd integers up to and including $(m-1)$.

Dividing both sides by $(k \log\log x)^{m/2}$, this gives that the weighted m-th moment of

$$\frac{\omega(n) - k \log\log x}{\sqrt{k \log\log x}}$$

is $(m-1)!! + o(1)$ if m is even and is $o(1)$ if m is odd. Recall that the m-th moment of a standard Gaussian random variable is $(m-1)!!$ if m is even and is 0 if m is odd. Thus, Theorem 1.1 implies the following weighted version of the Erdős–Kac Theorem.

Corollary 1.2. *Fix $k, m \in \mathbb{N}$. For any $\alpha \in \mathbb{R}$, we have*

$$\Big(\sum_{n \le x} \tau_k(n)\Big)^{-1} \sum_{\substack{n \le x \\ \omega(n) - k \log\log x \le \alpha \sqrt{k \log\log x}}} \tau_k(n) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-t^2/2}\, dt \tag{1.5}$$

as $x \to \infty$.

Throughout this paper, we will follow the ε-convention, where ε will always denote an arbitrarily small positive constant, but not necessarily the same one from one occurrence to the next. Our error terms are allowed to depend on ε.

2. Preliminaries

2.1. Averages of the k-fold divisor function

A well known result, due to Voronoi and Landau (see [23, Theorem 12.2]), states that for

$$\sum_{n\le x}\tau_k(n)=\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)\Big)+O\Big(x^{\frac{k-1}{k+1}+\varepsilon}\Big) \tag{2.1}$$

for $x\ge 1$ and $k\ge 2$. The leading order term of the residue is

$$\frac{x\,(\log x)^{k-1}}{(k-1)!},$$

while the lower order terms are proportional to $x(\log x)^{k-1-c}$ for integers c with $1\le c\le k-1$. We will need to sum $\tau_k(n)$ with n divisible by a fixed $a\in\mathbb{N}$. We prove such a result with a weaker error term than (2.1). This suffices for our application, as we only require a power savings in x/a below.

Lemma 2.1. *For $a\in\mathbb{N}$ and $x\ge a$, we have*

$$\sum_{\substack{n\le x\\ a|n}}\tau_k(n)=\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,a)\Big)+O\Big(\big(\frac{x}{a}\big)^{\frac{k+3}{k+6}+\varepsilon}\tau_k(a)\,M^{\omega(a)}\Big), \tag{2.2}$$

where

$$F(s,a):=\prod_{p^{\upsilon_p}||a}\Big(1-\Big(1-\frac{1}{p^s}\Big)^k\sum_{m=0}^{\upsilon_p-1}\frac{\tau_k(p^m)}{p^{ms}}\Big) \tag{2.3}$$

and

$$M:=\Big(\frac{\sqrt{2}+1}{\sqrt{2}-1}\Big)^6. \tag{2.4}$$

Though the function $F(s,a)$ depends on k, we suppress this in the notation since k is assumed to be fixed. Partial sums of divisor functions over arithmetic progressions have been extensively studied. In particular, Chace [4] has provided an asymptotic for the left hand side of (2.2). The form of our asymptotic, though it has a weaker error term, is better suited for our purposes. We postpone the proof of Lemma 2.1 to the end of this paper (see Section 5).

Let us look at the main term in (2.2). Writing out the Laurent series expansion of $\frac{x^s}{s}\zeta^k(s)F(s,a)$ around $s=1$, we obtain that the leading term of $\operatorname*{Res}_{s=1}\left(\frac{x^s}{s}\,\zeta^k(s)\,F(s,a)\right)$ is

$$F(1,a)\,\frac{x\,(\log x)^{k-1}}{(k-1)!}, \tag{2.5}$$

while the lower order terms are proportional to

$$x\,(\log x)^{k-1-c}\,\frac{d^j}{ds^j}\Big\{F(s,a)\Big\}\Big|_{s=1} \tag{2.6}$$

for integers j and c with $1\le c\le k-1$ and $0\le j\le c$. To study the behavior of $F(s,a)$ and its higher derivatives at $s=1$, we use its multiplicativity. For $a=p$, a prime, the expression in (2.3) takes the simpler form

$$F(s,p)=1-\Big(1-\frac{1}{p^s}\Big)^k.$$

When $s=1$, we can use binomial expansion to obtain the estimates

$$F(1,p)=1-\sum_{\ell=0}^{k}\binom{k}{\ell}\Big(\frac{-1}{p}\Big)^{\ell}=\frac{k}{p}+O\Big(\frac{1}{p^2}\Big) \tag{2.7}$$

and,

$$\frac{d^j}{ds^j}\Big\{F(s,p)\Big\}\Big|_{s=1}\ll\frac{(\log p)^j}{p} \tag{2.8}$$

for $j\ge 0$. Additionally, we also use Merten's estimates [18, Theorem 2.7], which are weaker than the prime number theorem, several times throughout the paper: for $x\ge 3$, we have

$$\sum_{p\le x}\frac{1}{p}=\log\log x+O(1) \tag{2.9}$$

and, for $j\ge 1$, we have

$$\sum_{p\le x}\frac{(\log p)^j}{p}\ll(\log x)^j. \tag{2.10}$$

2.2. Weighted average of $\omega(n)$

We want to show that, as n ranges over the integers below x, the average of $\omega(n)$ with respect to the weighted measure $\tau_k(n)$ is asymptotically $k \log\log x$. More precisely, we aim to prove that

$$\frac{\sum_{n\le x} \omega(n)\,\tau_k(n)}{\sum_{n\le x} \tau_k(n)} = k\log\log x + O(1). \tag{2.11}$$

With the rearrangement

$$\sum_{n\le x} \omega(n)\,\tau_k(n) = \sum_{n\le x}\Big(\sum_{p\mid n} 1\Big)\tau_k(n) = \sum_{p\le x}\sum_{\substack{n\le x\\ p\mid n}} \tau_k(n),$$

followed by an application of Lemma 2.1, the left-hand side of (2.11) is

$$\Big(\sum_{n\le x}\tau_k(n)\Big)^{-1}\sum_{p\le x}\left(\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,p)\Big) + O\Big(\Big(\frac{x}{p}\Big)^{\frac{k+3}{k+6}+\varepsilon}\tau_k(p)\,M^{\omega(p)}\Big)\right). \tag{2.12}$$

Using the estimate (2.1) and recalling the leading order term for the residue stated in (2.5), we see that the main term in (2.12) equals

$$\sum_{p\le x} F(1,p) = \sum_{p\le x}\frac{k}{p} + O(1) = k\log\log x + O(1),$$

where we have used (2.7) and Merten's result in (2.9).

Now, it suffices to show that the remaining terms in (2.12) are $O(1)$. First, to handle the error term, since k is fixed, we trivially have $\tau_k(p) = k = O(1)$ and $M^{\omega(p)} = M = O(1)$. Using the estimate (2.1), the contribution of error term in (2.12) is

$$\ll \frac{1}{x(\log x)^{k-1}}\sum_{p\le x}\Big(\frac{x}{p}\Big)^{\frac{k+3}{k+6}+\varepsilon} \ll 1.$$

Next, we handle the contribution of non-leading terms of the residue in (2.12). In view of (2.6), the remaining contribution of the residue is proportional to

$$\frac{1}{x\,(\log x)^{k-1}}\sum_{p\le x} x\,(\log x)^{k-1-c}\,\frac{d^j}{ds^j}\Big\{F(s,p)\Big\}\Big|_{s=1}$$

for $1\le c\le k-1$ and $0\le j\le c$. Using the estimate (2.8) followed by (2.10), the above expression is

$$\ll \frac{1}{(\log x)^c} \sum_{p \le x} \frac{(\log p)^c}{p} \ll 1.$$

This completes the proof that (2.11) follows from Lemma 2.1.

3. Reduction of Theorem 1.1

We will deduce Theorem 1.1 from the following technical proposition.

Proposition 3.1. *Define*

$$f_p(n) = \begin{cases} -F(1,p), & \text{if } p \nmid n, \\ 1 - F(1,p), & \text{if } p|n. \end{cases}$$

Let k and m be fixed natural numbers, and $3 < z \le x$. We have

$$\frac{\sum_{n \le x} \Big(\sum_{p \le z} f_p(n) \Big)^m \tau_k(n)}{\sum_{n \le x} \tau_k(n)} = \begin{cases} (m-1)!! \, \big(k \log\log z\big)^{m/2} + O\big((\log\log z)^{\frac{m-1}{2}}\big), & \text{if } m \text{ is even}, \\ O\big((\log\log z)^{\frac{m-1}{2}}\big), & \text{if } m \text{ is odd}, \end{cases} \tag{3.1}$$

where $(m-1)!!$ denotes the product of all odd integers up to and including $(m-1)$.

3.1. Deducing Theorem 1.1 from Proposition 3.1

Let $z = x^{\frac{1}{(k+6)m}}$. Recalling the estimate $\sum_{p \le x} F(1,p) = k \log\log x + O(1)$, we can write

$$\begin{aligned} \omega(n) - k \log\log x &= \sum_{p|n} 1 - \sum_{p \le x} F(1,p) + O(1) \\ &= \sum_{\substack{p \le z \\ p|n}} 1 + \sum_{\substack{p > z \\ p|n}} 1 - \sum_{p \le z} F(1,p) - \sum_{z < p \le x} F(1,p) + O(1) \\ &= \sum_{p \le z} f_p(n) + \sum_{\substack{p > z \\ p|n}} 1 - \sum_{z < p \le x} F(1,p) + O(1). \end{aligned}$$

For $n \le x$, we have

$$\sum_{\substack{p > z \\ p|n}} 1 = O(1)$$

because n can have at most $(k+6)m$ prime divisors between z and x, lest n will be larger than x. Furthermore,

$$\sum_{z<p\leq x} F(1,p) = k\log\log x - k\log\log z + O(1) = O(1),$$

thus yielding

$$\omega(n) - k\log\log x = \sum_{p\leq z} f_p(n) + O(1).$$

So, the binomial theorem gives

$$\frac{\sum\limits_{n\leq x}\Big(\omega(n)-k\log\log x\Big)^m \tau_k(n)}{\sum\limits_{n\leq x}\tau_k(n)} = \frac{\sum\limits_{n\leq x}\Big(\sum\limits_{p\leq z} f_p(n)\Big)^m \tau_k(n)}{\sum\limits_{n\leq x}\tau_k(n)} + O\left(\frac{\sum\limits_{n\leq x}\Big|\sum\limits_{p\leq z} f_p(n)\Big|^{m-1}\tau_k(n)}{\sum\limits_{n\leq x}\tau_k(n)}\right). \tag{3.2}$$

Now to deduce Theorem 1.1 from Proposition 3.1, it suffices to show that the size of error term in (3.2) is $\ll (\log\log z)^{\frac{m-1}{2}}$.

If $m-1$ is even, then the estimate (3.1) implies that the error term in (3.2) is $\ll (\log\log z)^{\frac{m-1}{2}}$. If $m-1$ is odd, then we use the Cauchy-Schwarz inequality to deduce that

$$\sum_{n\leq x}\Big|\sum_{p\leq z} f_p(n)\Big|^{m-1}\tau_k(n) \leq \Big(\sum_{n\leq x}\Big(\sum_{p\leq z} f_p(n)\Big)^{m-2}\tau_k(n)\Big)^{\frac{1}{2}}\Big(\sum_{n\leq x}\Big(\sum_{p\leq z} f_p(n)\Big)^{m}\tau_k(n)\Big)^{\frac{1}{2}}.$$

Again using (3.1), we see that the error term in (3.2) is $\ll (\log\log z)^{\frac{m-1}{2}}$.

Combining the above estimates, this shows that Theorem 1.1 is a consequence of Proposition 3.1.

4. Proof of Proposition 3.1

For $f_p(n)$ defined in Proposition 3.1 and $r\in\mathbb{N}$, define $f_r(n) := \prod_{p^\alpha || r} f_p(n)^\alpha$. Thus $f_r(n)$ is totally multiplicative in r. Then, we can write

$$\sum_{n\leq x}\Big(\sum_{p\leq z} f_p(n)\Big)^m \tau_k(n) = \sum_{p_1,p_2,\ldots,p_m\leq z}\ \sum_{n\leq x} f_{p_1p_2\cdots p_m}(n)\,\tau_k(n),$$

which allows us to write the left-hand side of (3.1) as

$$\sum_{p_1,p_2,\ldots,p_m\le z}\frac{\sum_{n\le x} f_{p_1p_2\cdots p_m}(n)\,\tau_k(n)}{\sum_{n\le x}\tau_k(n)}. \tag{4.1}$$

We now evaluate (4.1) to prove Proposition 3.1.

Writing $r=p_1p_2\cdots p_m$, let us consider $\sum_{n\le x} f_r(n)\,\tau_k(n)$. Since there are m primes p_i and each $p_i\le z$, where $z=x^{\frac{1}{(k+6)m}}$, we only need to consider the range $r\le x^{\frac{1}{k+6}}$. Let R denote the square-free part of r. That is, $R=\prod_{p^\alpha\|r} p$. Notice that, by definition, if $a=(n,R)$, then $f_r(n)=f_r(a)$. Thus, we can write

$$\sum_{n\le x} f_r(n)\,\tau_k(n)=\sum_{a|R} f_r(a)\sum_{\substack{n\le x\\ (n,R)=a}}\tau_k(n)=\sum_{a|R} f_r(a)\sum_{\substack{n\le x\\ a|n\\ (R/a,n/a)=1}}\tau_k(n).$$

Using the identity $\sum_{b|n,b|m}\mu(b)=1$ if $(n,m)=1$ and equals 0 otherwise, it follows that

$$\begin{aligned}
&\sum_{a|R} f_r(a)\sum_{\substack{n\le x\\ a|n\\ (R/a,n/a)=1}}\tau_k(n)\\
&\quad=\sum_{ab|R} f_r(a)\,\mu(b)\sum_{\substack{n\le x\\ ab|n}}\tau_k(n)\\
&\quad=\sum_{ab|R} f_r(a)\,\mu(b)\left(\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,ab)\Big)+O\Big(\Big(\frac{x}{ab}\Big)^{\frac{k+3}{k+6}+\varepsilon}\tau_k(ab)\,M^{\omega(ab)}\Big)\right),
\end{aligned}$$

where the last equality follows from Lemma 2.1. Since $M^{\omega(ab)}\le\tau_{\lceil M\rceil}(ab)\le(ab)^\varepsilon$, which uses that ab is square-free, and $\tau_k(ab)\ll(ab)^\varepsilon$, the error term above is $O(x^{\frac{k+3}{k+6}+\varepsilon})$. When this error term is summed for all $ab|R$, the total contribution is

$$\ll\sum_{r\le x^{\frac{1}{k+6}}} x^{\frac{k+3}{k+6}+\varepsilon}\ll x^{\frac{k+4}{k+6}+\varepsilon}.$$

Thus, using the estimate (2.1), we deduce that

$$\frac{\sum_{n\le x} f_r(n)\,\tau_k(n)}{\sum_{n\le x}\tau_k(n)}=\frac{(k-1)!}{x(\log x)^{k-1}}\sum_{ab|R} f_r(a)\,\mu(b)\left(\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\,\zeta^k(s)\,F(s,ab)\Big)\right)+O(x^{\frac{-2}{k+6}+\varepsilon}). \tag{4.2}$$

4.1. Main term of (4.1)

Recalling the leading order term of the residue stated in (2.5), we will see that the main term in (4.2) is $\sum_{ab|R} f_r(a)\mu(b)F(1,ab)$. For convenience, let us give this main term a name by defining

$$G(r) := \sum_{ab|R} f_r(a)\,\mu(b)\,F(1,ab).$$

It is easy to see that $G(r)$ is multiplicative in r and therefore

$$G(r) = \prod_{p^\alpha || r} G(p^\alpha).$$

Notice that for any prime p, we have

$$\begin{aligned} G(p) &= (\text{contribution of } ab = 1) + (\text{contribution of } a = p, b = 1) \\ &\quad + (\text{contribution of } a = 1, b = p) \\ &= -F(1,p) + (1 - F(1,p))\,F(1,p) - (-F(1,p))\,F(1,p) \\ &= 0. \end{aligned}$$

This is an important observation because it implies that $G(r)$ is supported on square-full integers. We also note that for $\alpha = 2$, we have

$$\begin{aligned} G(p^2) &= \big(F(1,p)\big)^2 + (1 - F(1,p))^2 F(1,p) - \big(-F(1,p)\big)^2 F(1,p) \\ &= \big(F(1,p)\big)\big(1 - F(1,p)\big) \geq 0 \end{aligned}$$

as $0 < F(1,p) < 1$. For $\alpha \geq 2$, using the estimate (2.7), we get

$$G(p^\alpha) = \frac{k}{p} + O\Big(\frac{1}{p^2}\Big). \tag{4.3}$$

Since $G(r)$ is supported over square-full integers r, the contribution to (4.2) from the leading term of the residue (2.5) is

$$\sum_{\substack{p_1, p_2, \ldots, p_m \leq z \\ p_1 p_2 \cdots p_m \text{ square-full}}} G(p_1 p_2 \cdots p_m).$$

Denote $q_1 < q_2 < \ldots < q_t$ to be the distinct primes in the list $p_1, p_2, \ldots, p_m$. Thus $p_1 p_2 \cdots p_m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_t^{\alpha_t}$. Since $p_1 p_2 \cdots p_m$ is square-full, we have $\alpha_i \geq 2$ and $t \leq m/2$. By multiplicativity, the expression above can be written as

$$\sum_{t\le m/2}\ \sum_{q_1<q_2<\ldots<q_t\le z}\ \sum_{\substack{\alpha_1,\ldots,\alpha_t\ge 2\\ \sum\alpha_i=m}} \frac{m!}{\alpha_1!\ldots\alpha_t!}G(q_1^{\alpha_1})\cdots G(q_t^{\alpha_t}).$$

When m is even, we have a term $t=m/2$, where all $\alpha_i=2$, that yields the expected Gaussian moment for the following reason. Using the estimate (4.3), the contribution of this term is

$$\begin{aligned}&\frac{m!}{2^{m/2}}\sum_{q_1<q_2<\ldots<q_{m/2}\le z}\prod_{i=1}^{m/2}\left(\frac{k}{q_i}+O\Big(\frac{1}{q_i^2}\Big)\right)\\ &=\frac{m!}{2^{m/2}(m/2)!}\sum_{\substack{q_1,q_2,\ldots,q_{m/2}\le z\\ \text{distinct}}}\prod_{i=1}^{m/2}\left(\frac{k}{q_i}+O\Big(\frac{1}{q_i^2}\Big)\right).\end{aligned}$$

Dropping the condition that the primes q_i need to be distinct, we note that the sum is clearly bounded from above by

$$\left(\sum_{q\le z}\frac{k}{q}+O\Big(\frac{1}{q^2}\Big)\right)^{m/2}=\big(k\log\log z\big)^{m/2}+O\Big((\log\log z)^{\frac{m}{2}-1}\Big).$$

If we consider $q_1,\ldots,q_{m/2-1}$ as given, then the sum over $q_{m/2}$ is at least

$$\sum_{\pi_{m/2}\le q\le z}\frac{k}{q}+O\Big(\frac{1}{q^2}\Big),$$

where π_n denotes the n-th prime. This is because $q_{m/2}$ must be distinct from the other $m/2-1$ primes, and to get this lower bound we don't allow it to be any of the smallest $m/2-1$ primes, for $\frac{k}{q}+O(\frac{1}{q^2})$ increases as q decreases. By the same argument we get a lower bound for the sum over each q_i. In this way, we see that the sum over $q_1,q_2,\ldots,q_{m/2}$ distinct is bounded from below by

$$\left(\sum_{\pi_{m/2}\le q\le z}\frac{k}{q}+O\Big(\frac{1}{q^2}\Big)\right)^{m/2}=\big(k\log\log z\big)^{m/2}+O\Big((\log\log z)^{\frac{m}{2}-1}\Big).$$

To derive the second expression from the first, keep in mind that $m=O(1)$. Thus, the upper and lower bounds are the same up to a $m/2-1$ power of $\log\log z$. So we can conclude that the contribution of the term with $t=m/2$ is

$$\frac{m!}{2^{m/2}(m/2)!}\big(k\log\log z\big)^{m/2}+O\Big((\log\log z)^{\frac{m}{2}-1}\Big). \tag{4.4}$$

Notice that the main term above is the main term in (3.1), as desired.

4.2. Error terms in (4.1)

Continuing with the notation above, we first consider the contribution of the terms with $t < m/2$. We use $G(q_i^{\alpha_i}) \ll 1/q_i$ and get that the contribution of such terms is

$$\ll \sum_{q_1<q_2<\ldots<q_t\le z} \frac{1}{q_1\cdots q_t} \ll \left(\sum_{q\le z}\frac{1}{q}\right)^t \ll (\log\log z)^t \ll (\log\log z)^{\frac{m}{2}-1}. \tag{4.5}$$

The remaining terms in (4.1) are the non-leading terms of the residue and the error term in (4.2). The contribution of the error term is

$$\ll \sum_{p_1,p_2,\ldots,p_m\le z} x^{\frac{-2}{k+6}+\varepsilon} \ll \pi(z)^m\, x^{\frac{-2}{k+6}+\varepsilon},$$

where $\pi(z)$ is the prime counting function. Recalling that $z = x^{\frac{1}{m(k+6)}}$, we get $\pi(z)^m \ll z^m = x^{\frac{1}{k+6}}$. So, the error term above is

$$\ll x^{\frac{-1}{k+6}+\varepsilon} \ll 1. \tag{4.6}$$

Next we handle the contribution of the non-leading terms of the residue. In view of (2.6), the contribution of such terms is bounded by

$$\sum_{\substack{1\le c\le k-1\\ 0\le j\le c}} \left| \frac{1}{(\log x)^c} \sum_{p_1,p_2,\ldots,p_m\le z} \frac{d^j}{ds^j}\Big\{G(s,p_1p_2\cdots p_m)\Big\}\Big|_{s=1}\right|, \tag{4.7}$$

where we define

$$G(s,r) := \sum_{ab|R} f_r(a)\,\mu(b)\,F(s,ab).$$

Note that $G(1,r) = G(r)$. As before, let $p_1p_2\cdots p_m = q_1^{\alpha_1}q_1^{\alpha_1}\cdots q_t^{\alpha_t}$ for distinct primes $q_1 < q_2 < \ldots < q_t$ in the list $p_1, p_2, \ldots, p_m$ and, using multiplicativity, write the expression within absolute values above as

$$\frac{1}{(\log x)^c}\sum_{t\le m}\sum_{q_1<q_2<\ldots<q_t\le z}\sum_{\substack{\alpha_1,\ldots,\alpha_t\ge 1\\ \sum\alpha_i=m}} \frac{m!}{\alpha_1!\cdots\alpha_t!}\frac{d^j}{ds^j}\Big\{G(s,q_1^{\alpha_1})\cdots G(s,q_t^{\alpha_t})\Big\}\Big|_{s=1}.$$

Using the product rule for differentiation, this can be further expressed as

$$\frac{1}{(\log x)^c}\sum_{t\le m}\sum_{q_1<q_2<\ldots<q_t\le z}\sum_{\substack{\alpha_1,\ldots,\alpha_t\ge 1\\ \sum\alpha_i=m}} \frac{m!}{\alpha_1!\cdots\alpha_t!}$$

$$\sum_{\substack{\beta_1,\beta_2,\ldots,\beta_t\geq 0\\ \beta_1+\beta_2+\ldots+\beta_t=j}} \frac{j!}{\beta_1!\cdots\beta_t!}\prod_{i=1}^{t}\frac{d^{\beta_i}}{ds^{\beta_i}}\Big\{G(s,q_i^{\alpha_i})\Big\}\Big|_{s=1}.$$

Here, the sum over β_is counts all possible ways the $G(s,q_i^{\alpha_i})$ terms can be differentiated using the product rule. Some β_i values can be 0, which represents $G(s,q_i^{\alpha_i})$ that are not differentiated. If we have a case where $\beta_i = 0$ and $\alpha_i = 1$, then the whole sum collapses to 0 because we will have a factor $G(1,q_i) = 0$ in the product (recall that $G(r)$ is supported on square-full r). Therefore, every occurrence of $G(s,q_i)$ needs to be differentiated. Recall by (4.3) that $G(s,q^\alpha) \ll \frac{1}{q}$ and observe that (2.8) implies that

$$\frac{d^{\beta_i}}{ds^{\beta_i}}\Big\{G(s,q^\alpha)\Big\}\Big|_{s=1} \ll \frac{(\log q)^{\beta_i}}{q}.$$

Using these estimates, we obtain that (4.7) is

$$\ll \sum_{\substack{1\leq c\leq k-1\\ 0\leq j\leq c}}\sum_{t\leq m}\frac{1}{(\log x)^c}\sum_{\substack{\beta_1,\beta_2,\ldots,\beta_t\geq 0\\ \beta_1+\beta_2+\ldots+\beta_t=j}}\prod_{i=1}^{i=t}\Big(\sum_{q\leq z}\frac{(\log q)^{\beta_i}}{q}\Big).$$

Using (2.10), we get that this is bounded by

$$\sum_{\substack{1\leq c\leq k-1\\ 0\leq j\leq c}}\sum_{t\leq m}\frac{1}{(\log x)^c}(\log z)^j(\log\log z)^{|\{1\leq i\leq t:\ \beta_i=0\}|} \tag{4.8}$$

Here, $|\{1\leq i\leq t:\ \beta_i=0\}|$ is the number of terms $G(s,q_i^{\alpha_i})$ which are not differentiated. Since all $G(s,q_i^{\alpha_i})$ that remain undifferentiated have $\alpha_i \geq 2$, lest the entire expression vanishes, and not all $\alpha_i = 2$, since those terms were treated in the previous section. Hence, we must have

$$|\{1\leq i\leq t:\ \beta_i=0\}| \leq \frac{m-1}{2}.$$

Therefore, we get that the expression in (4.8) is bounded by $(\log\log z)^{\frac{m-1}{2}}$. This completes the proof of Proposition 3.1.

5. Proof of Lemma 2.1

Our proof of Lemma 2.1 involves a standard application of Perron's formula applied to the Dirichlet series associated with the sum on the left-hand side of (2.2). Making the substitution $n = ab$, we can express this associated Dirichlet series as

$$\sum_{\substack{n=1\\ a|n}}^{\infty}\frac{\tau_k(n)}{n^s} = \frac{1}{a^s}\sum_{b=1}^{\infty}\frac{\tau_k(ab)}{b^s} = \frac{1}{a^s}\Bigg(\prod_{p^{\upsilon_p}||a}\sum_{m=0}^{\infty}\frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}}\Bigg)\Bigg(\prod_{p\nmid a}\sum_{m=0}^{\infty}\frac{\tau_k(p^m)}{p^{ms}}\Bigg), \tag{5.1}$$

where the second equality follows from using the multiplicativity of $\tau_k(n)$ to express the sum as an Euler product. Using the well-known Dirichlet series of $\zeta^k(s)$ and its corresponding Euler product,

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}}, \tag{5.2}$$

we can complete the product over primes in (5.1) to obtain Euler product of $\zeta^k(s)$ and write

$$\sum_{\substack{n=1 \\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \frac{\zeta^k(s)}{a^s} \prod_{p^{\upsilon_p}||a} \left(\sum_{m=0}^{\infty} \frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}} \Big/ \sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} \right). \tag{5.3}$$

Now, writing the sum in numerator on the right-hand side as

$$\sum_{m=0}^{\infty} \frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}} = p^{s\upsilon_p} \Big(\sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} - \sum_{m=0}^{\upsilon_p - 1} \frac{\tau_k(p^m)}{p^{ms}} \Big)$$

and using the identity

$$\sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} = \Big(\frac{p^s}{p^s - 1} \Big)^k, \tag{5.4}$$

which follows by taking the Euler product of $\zeta(s)$ on the left hand side of (5.2), we can reduce (5.3) to

$$\sum_{\substack{n=1 \\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta^k(s) \prod_{p^{\upsilon_p}||a} \Big(1 - \Big(1 - \frac{1}{p^s} \Big)^k \sum_{m=0}^{\upsilon_p - 1} \frac{\tau_k(p^m)}{p^{ms}} \Big).$$

With the definition of $F(s, a)$ stated in (2.3), this can be expressed as

$$\sum_{\substack{n=1 \\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta^k(s)\, F(s, a).$$

Note this Dirichlet series converges absolutely for $\mathrm{Re}(s) > 1$.

An application of Perron's formula (see [20, Part II. §2.1]) gives

$$\sum_{\substack{n \le x \\ a|n}} \tau_k(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s)^k\, F(s, a)\, x^s \frac{ds}{s} \; + \; O\Bigg(\sum_{\substack{n \ge 1 \\ a|n}} \Big(\frac{x}{n} \Big)^c \frac{\tau_k(n)}{T |\log \frac{x}{n}|} \Bigg), \tag{5.5}$$

where we choose c and T such that $c > 1$ and T is large. First, let us handle the error term in (5.5). Making the substitution $n = ab$ and using inequality $\tau_k(ab) \le \tau_k(a)\tau_k(b)$, we can write the expression in the error term as

$$\left(\frac{x}{a}\right)^c \frac{\tau_k(a)}{T} \sum_{b=1}^{\infty} \frac{\tau_k(b)}{b^c \left|\log\left(\frac{x/a}{b}\right)\right|}.$$

We estimate this error by splitting the range of summation into two pieces. The terms where $b < \frac{x/a}{2}$ or $b > \frac{3x/a}{2}$ contribute an amount that is bounded by

$$\left(\frac{x}{a}\right)^c \frac{\tau_k(a)}{T} \sum_{\substack{b<\frac{x/a}{2} \\ b>\frac{3x/a}{2}}} \frac{\tau_k(b)}{b^c \left|\log\left(\frac{x/a}{b}\right)\right|} \ll \left(\frac{x}{a}\right)^c \frac{\tau_k(a)}{T} \sum_{b=1}^{\infty} \frac{\tau_k(b)}{b^c} = \left(\frac{x}{a}\right)^c \frac{\zeta^k(c)\,\tau_k(a)}{T}$$
$$\ll \left(\frac{x}{a}\right)^c \frac{\tau_k(a)}{T\,(c-1)^k}.$$

Here we used the fact that $\zeta(c) \ll (1-c)^{-1}$ for $c > 1$. When $\frac{x/a}{2} \le b \le \frac{3x/a}{2}$, we have $b^c \asymp (x/a)^c$ and $\tau_k(b) \ll b^{\varepsilon/2} \ll (x/a)^{\varepsilon/2}$. So, the size of error is

$$\left(\frac{x}{a}\right)^c \tau_k(a) \sum_{\frac{x/a}{2}<b<\frac{3x/a}{2}} \frac{\tau_k(b)}{b^c T \left|\log\left(\frac{x/a}{b}\right)\right|} \ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{\varepsilon/2} \sum_{\frac{x/a}{2}<b<\frac{3x/a}{2}} \frac{1}{\left|\log\left(\frac{x/a}{b}\right)\right|}. \tag{5.6}$$

To evaluate the rightmost sum above, we note that $b = \lfloor \frac{x}{a} \rfloor + \nu$, where $-0.5x/a \le \nu \le 0.5x/a$. As usual, we let $\lfloor \frac{x}{a} \rfloor$ and $\{\frac{x}{a}\}$ denote the integer part and the fractional part of $\frac{x}{a}$, respectively, and we note that

$$\left|\log \frac{x/a}{b}\right| = \left|\log \frac{\lfloor \frac{x}{a} \rfloor + \{x/a\}}{\lfloor \frac{x}{a} \rfloor + \nu}\right| = \left|\log\left(1 - \frac{\nu - \{x/a\}}{\lfloor \frac{x}{a} \rfloor + \nu}\right)\right| \asymp \frac{|\nu| a}{x}.$$

Thus, (5.6) is

$$\ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{\varepsilon/2} \sum_{|\nu|\le 0.5x/a} \frac{x}{a|\nu|} \ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{1+\varepsilon/2} \log\left(\frac{x}{a}\right) \ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{1+\varepsilon},$$

using the estimate $\log\left(\frac{x}{a}\right) \ll \left(\frac{x}{a}\right)^{\varepsilon/2}$. Letting $c = 1 + \varepsilon$, we conclude that

$$\sum_{\substack{n\le x \\ a|n}} \tau_k(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta(s)^k F(s,a)\, x^s \frac{ds}{s} + O\left(\frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{1+\varepsilon}\right). \tag{5.7}$$

Next we evaluate the integral on the right-hand side of (5.7) using the residue theorem. Observing that the only singularity of the integrand inside the (positively oriented) rectangle $\mathcal{R}$ with vertices $c-iT$, $c+iT$, $\frac{1}{2}+iT$, and $\frac{1}{2}-iT$ is at $s=1$, the calculus of residues gives

$$\frac{1}{2\pi i}\int\limits_{\mathcal{R}}\zeta(s)^k\,F(s,a)\,x^s\,\frac{ds}{s}=\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,a)\Big).$$

The integral along the right-hand edge of $\mathcal{R}$ corresponds to the integral in (5.7), so it suffices to estimate the contribution of the integrals along the left-hand edge and horizontal portions of the contour. To bound these integrals, we recall the classical (subconvexity) estimate

$$\zeta\big(\tfrac{1}{2}+it\big)\ll(|t|+1)^{\frac{1}{6}},\qquad\text{for } t>0. \tag{5.8}$$

This estimate and the Phragmén–Lindelöf principle imply that

$$\zeta(\sigma+it)\ll(|t|+1)^{\frac{1}{6}(c-\sigma)/(c-\frac{1}{2})} \tag{5.9}$$

uniformly for $\sigma\in[\frac{1}{2},c]$ and $t\in[-T,T]$. We also need an estimate of $F(s,a)$ in this region. Notice that $F(s,a)$ can be expressed as the right-hand side of (5.3) divided by $\zeta^k(s)$. Thus, we have

$$|F(s,a)|=\frac{1}{a^\sigma}\prod_{p^{\upsilon_p}||a}\left|\Big(\sum_{m=0}^{\infty}\frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}}\Big/\sum_{m=0}^{\infty}\frac{\tau_k(p^m)}{p^{ms}}\Big)\right|,$$

where $\sigma=\operatorname{Re}(s)$. Using the inequality $\tau_k(p^{m+\upsilon_p})\le\tau_k(p^m)\,\tau_k(p^{\upsilon_p})$ and (5.4), we derive that

$$\Big|\sum_{m=0}^{\infty}\frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}}\Big|\le\tau_k(p^{\upsilon_p})\sum_{m=0}^{\infty}\frac{\tau_k(p^m)}{p^{m\sigma}}=\tau_k(p^{\upsilon_p})\Big(\frac{p^\sigma}{p^\sigma-1}\Big)^k.$$

The identity (5.4) gives the bound of remaining sum as well, namely

$$\Big|\Big(\sum_{m=0}^{\infty}\frac{\tau_k(p^m)}{p^{ms}}\Big)^{-1}\Big|=\Big|\Big(\frac{p^s-1}{p^s}\Big)^k\Big|\le\Big(\frac{p^\sigma+1}{p^\sigma}\Big)^k.$$

Notice that $\big((p^\sigma+1)/(p^\sigma-1)\big)^k\le\big((\sqrt{2}+1)/(\sqrt{2}-1)\big)^k=M^{k/6}$ for $\sigma\ge\frac{1}{2}$, where M was defined in (2.4). So, combining these inequalities yields

$$|F(s,a)|\le\frac{1}{a^\sigma}\prod_{p^{\upsilon_p}||a}\tau_k(p^{\upsilon_p})\Big(\frac{p^\sigma+1}{p^\sigma-1}\Big)^k\le\frac{\tau_k(a)\,M^{\frac{k}{6}\omega(a)}}{a^\sigma},\qquad\text{for }\sigma\ge\tfrac{1}{2}. \tag{5.10}$$

Now, using the estimates for $\zeta(s)$ and $F(s,a)$ stated in (5.8), (5.9), and (5.10), respectively, the integrals along the horizontal edges of $\mathcal{R}$ can be estimated as

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}\pm iT}^{c\pm iT}\zeta^k(s)\,F(s,a)\,x^s\,\frac{ds}{s}\right| = O\bigg(\tau_k(a)\,M^{\frac{k}{6}\omega(a)}\int_{\frac{1}{2}}^{c}\Big(\frac{x}{a}\Big)^{\sigma}T^{\frac{k}{6}(c-\sigma)/(c-\frac{1}{2})-1}d\sigma\bigg)$$
$$= O\Big(T^{-1}\Big(\frac{x}{a}\Big)^{c}\tau_k(a)M^{\frac{k}{6}\omega(a)}\Big)$$
$$+ O\Big(T^{\frac{k}{6}-1}\Big(\frac{x}{a}\Big)^{\frac{1}{2}}\tau_k(a)M^{\frac{k}{6}\omega(a)}\Big),$$

since the maximum of the integrand occurs at one of the endpoints. Similarly, the integral along the left-hand side of $\mathcal{R}$ can be estimated as

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta^k(s)\,F(s,a)\,x^s\,\frac{ds}{s}\right| = O\bigg(\Big(\frac{x}{a}\Big)^{\frac{1}{2}}\tau_k(a)\,M^{\frac{k}{6}\omega(a)}\int_{-T}^{T}|\zeta(\tfrac{1}{2}+it)|^k\,\frac{dt}{|t|+1}\bigg)$$
$$= O\bigg(\Big(\frac{x}{a}\Big)^{\frac{1}{2}}\tau_k(a)\,M^{\frac{k}{6}\omega(a)}\int_{-T}^{T}(|t|+1)^{\frac{k}{6}-1}dt\bigg)$$
$$= O\bigg(\Big(\frac{x}{a}\Big)^{\frac{1}{2}}T^{\frac{k}{6}}\,\tau_k(a)\,M^{\frac{k}{6}\omega(a)}\bigg).$$

Collecting estimates, we have shown that

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta(s)^k\,F(s,a)\,x^s\,\frac{ds}{s} = \operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,a)\Big) + O\bigg(\frac{\tau_k(a)\,M^{\frac{k}{6}\omega(a)}}{T}\Big(\frac{x}{a}\Big)^{c}\bigg)$$
$$+ O\Big(T^{\frac{k}{6}}\Big(\frac{x}{a}\Big)^{\frac{1}{2}}\tau_k(a)\,M^{\frac{k}{6}\omega(a)}\Big)$$

Inserting this estimate into (5.7) and recalling that $c = 1+\varepsilon$, we find that

$$\sum_{\substack{n\le x\\ a|n}}\tau_k(n) = \operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,a)\Big) + O\bigg(\frac{\tau_k(a)\,M^{\frac{k}{6}\omega(a)}}{T}\Big(\frac{x}{a}\Big)^{1+\varepsilon}\bigg)$$
$$+ O\Big(T^{\frac{k}{6}}\Big(\frac{x}{a}\Big)^{\frac{1}{2}}\tau_k(a)\,M^{\frac{k}{6}\omega(a)}\Big).$$

Up to a factor of ε, the error terms are (essentially) minimized by choosing

$$T = \Big(\frac{x}{a}\Big)^{\frac{3}{k+6}}M^{\frac{-k}{k+6}\omega(a)},$$

which finishes the proof of Lemma 2.1.

Remark. Although we used the subconvexity estimate (5.8), we do not actually need anything nontrivial. Instead, we could have used any estimate of the form $\zeta(\frac{1}{2}+it) \ll (|t|+1)^A$ for a fixed $A>0$. This would lead to a weaker result, albeit still a power savings in x/a. Moreover, it is likely that a version of Lemma 2.1 can be proved in an elementary manner with a power savings in x/a and perhaps a different formula for the main term. We chose to use Perron's formula for simplicity.

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